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LETTER TO THE EDITOR

Disorder solution of a general checkerboard Ising model in a field and validity of the decimation approach

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Abstract. We examine a problem arising in the determination of the disorder solution using the method of exact decimation for spin models in a field. The problem concerns the validity of the change of boundary conditions, a crucial step essential to the decimation approach. We study this validity question for a general checkerboard Ising lattice which has crossing and multispin interactions as well as a magnetic field. Our study leads to an equation determining the disorder solution and a sufficient condition ensuring the validity of the solution. The nearest-neighbour model is re-examined in light of these discussions.

Recently, in an application of the method of exact decimation (Jaekel and Maillard 1985a, Wu 1985), Jaekel and Maillard (1985b) obtained the disorder solution for the Ising and Potts models with a field on the checkerboard lattice. The purpose of this letter is to identify a difficulty inherent in applying the decimation approach to problems with a field, and propose a solution. The question, which was not analysed by Jaekel and Maillard (1985b), is whether the bulk partition function is affected by the change of boundary conditions introduced in a crucial step of the decimation approach. We use the Ising model on a general checkerboard lattice as an example, and conclude that there is no simple way to ascertain the general validity of the disorder solution. However, a sufficient condition can be written down, and the Jaekel–Maillard solution is re-examined in light of these discussions. We also extend the disorder solution for the checkerboard Ising model to include both crossing and multispin interactions.

Consider N Ising spins on a square lattice with a Hamiltonian having a checkerboard-type symmetry as shown in figure 1. The four spins σ_1, σ_2, σ_3 and σ_4 surrounding each shaded square in figure 1 interact with a spin-reversal invariant interaction, which can be written, in the most general case, as§

$$E(\sigma_1\sigma_2\sigma_3\sigma_4) = -J_0 - J_1\sigma_1\sigma_2 - J_1'\sigma_3\sigma_4 - J_2\sigma_4\sigma_1 - J_2'\sigma_2\sigma_3 - J\sigma_1\sigma_3 - J'\sigma_2\sigma_4 - J_4\sigma_1\sigma_2\sigma_3\sigma_4. \quad (1)$$

The interactions J are depicted in figure 2. In addition, we assume the presence of an external magnetic field H. Our goal is to evaluate the partition function for the Ising lattice. Jaekel and Maillard (1985b) considered the special case $J = J' = J_4 = 0$,

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§ The restriction to spin-reversal invariant interactions is not essential to our conclusions.

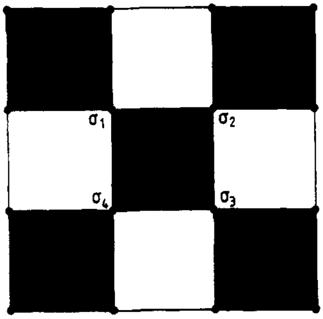

Figure 1. General checkerboard lattice. Each shaded square consists of interactions shown in figure 2.

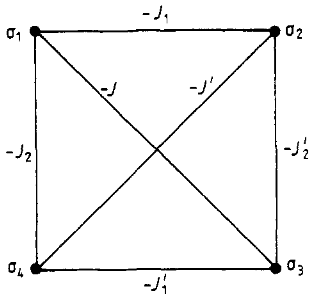

Figure 2. Interaction (1) contained in a shaded square in figure 1. (The four-spin interaction J_4 is not shown.)

and Wu (1985) has obtained the disorder solution in the triangular limit $J_1 = \infty$ and with three-spin interactions.

We assume a boundary condition which is periodic in the horizontal direction and open in the vertical direction. (The lattice is wound on a cylinder.) Further, we remove all $-J_1'$ interactions along the upper boundary and change the magnetic field applied to the spins on the upper boundary to alternate new values H_1 and H_2. The removal of the $-J_1'$ interactions permits us to carry out the spin summations over σ_1 and σ_2 (cf figure 1) for each shaded square in the first row. Now we require the summations to yield

$$\sum_{\sigma_1\sigma_2} w(\sigma_1\sigma_2\sigma_3\sigma_4)\exp(L_1\sigma_1 + L_2\sigma_2) = F\exp[(L_1 - L)\sigma_3 + (L_2 - L)\sigma_4] \quad (2)$$

where $L = H/kT$, $L_i = H_i/KT$, and

$$w(\sigma_1\sigma_2\sigma_3\sigma_4) = \exp[-E(\sigma_1\sigma_2\sigma_3\sigma_4)/kT]. \quad (3)$$

The summation (2) decimates all shaded squares in the first row, leaving a lattice which is an exact copy of the original one except that it has one less row. Furthermore, (2) implies that the new boundary spins again have alternate fields L_1 and L_2, and we can repeat the decimation process by summing over the new boundary spins. Continuing in this fashion, we eventually decimate all spins except those in the last row. Now each decimated shaded square contributes a factor F to the partition function Z, and the contribution from the last row of spins can be neglected in the bulk limit.

We then obtain the following expression for the partition function per site

$$\kappa = \lim_{N\to\infty} Z^{1/N} = |F|^{1/2}, \tag{4}$$

where we have allowed the possibility that F may be negative.

Explicitly, (2) is a set of four equations

$$\begin{pmatrix} w_1 & w_3 & w_5 & w_7 \\ w_3 & w_1 & w_7 & w_5 \\ w_6 & w_8 & w_2 & w_4 \\ w_8 & w_6 & w_4 & w_2 \end{pmatrix} \begin{pmatrix} x_1x_2 \\ 1 \\ x_1 \\ x_2 \end{pmatrix} = F \begin{pmatrix} x_1x_2\,\mathrm{e}^{-2L} \\ \mathrm{e}^{2L} \\ x_1 \\ x_2 \end{pmatrix} \tag{5}$$

where

$$x_i = \mathrm{e}^{2L_i}, \qquad i = 1, 2, \tag{6}$$

and

$$\begin{aligned} &w_1 = w(++++), \quad &&w_2 = w(+-+-), \quad &&w_3 = w(++--), \quad &&w_4 = (+--+) \\ &w_5 = w(+-++), \quad &&w_6 = w(+++-), \quad &&w_7 = w(-+++), \quad &&w_8 = (++-+). \end{aligned} \tag{7}$$

From (5) we can solve, in principle, for the four unknowns x_1, x_2, F and L in terms of the Boltzmann weights w. The partition function is then given by (4) with L given by the solution of (5).

Before examining (5) for solutions, we consider the validity of the change of boundary conditions used in arriving at (4), a question not analysed by Jaekel and Maillard (1985b). The crux of the matter is that we wish to ascertain that the change of boundary conditions does not affect the bulk partition function so that (4) is the true solution. Certainly, the change of boundary fields to new values L_1 and L_2 (and the deletion of the boundary J_1' interactions) will not affect the bulk properties as long as

$$x_i > 0, \qquad i = 1, 2. \tag{8}$$

For, in this case, the LHS of (5) is strictly positive and the usual proof of boundary independence (see, e.g., Fisher 1964) can be carried through. Therefore, (8) is certainly a sufficient condition to ensure the validity of the disorder solution (4).

However, it is not clear whether (8) is always a required condition. An explicit example is the triangular limit of $K_1 = \infty$ considered by Wu (1985). In the present notation we have $w_2 = w_4 = w_5 = w_7 = 0$ and it is sufficient for the positivity of the LHS of (5) to require only $x_1x_2 > 0$, a condition obtained more directly by Wu (1985). Indeed, in this case (5) admits solutions $x_i < 0$, $F < 0$, $\mathrm{e}^{2L} < 0$; the latter inequality being permitted since the partition function is now a polynomial of e^{4L}. It is clear then, for K_1 large at least, some non-positive x_i's are valid solutions as a consequence of a global nature beyond the description of (5). More generally, (5) admits solutions $x_i < 0$, for real F and L, provided that we permit both sides of some equations in (5) to be negative. While this makes some Boltzmann factors negative, the bulk partition function may still very well remain unchanged. It appears that it would be a difficult task to determine *a priori* if a particular solution of (5) is valid, other than comparing the resulting κ with the exact solution, if known. The best one can say, it seems, is that (8) provides a sufficient criterion for testing the decimation approach. Consequently, extreme caution must be exercised in using the decimation approach for problems with a field.

We now examine (5). The solution of (5) in the most general case involves solving a high degree algebraic equation, and is therefore complicated. However, one can readily eliminate x_1 and x_2 by taking the determinant of (5), and this leads to the following relation between L and F

$$2\cosh 2L = \frac{F^4 - 2w_2F^3 + (w_1^2 + w_2^2 - w_3^2 - w_4^2)F^2 - 2D_1F + D_2}{w_1F^3 + (w_5w_6 + w_7w_8 - 2w_1w_2)F^2 - D_3F} \tag{9}$$

where

$$D_1 = \begin{vmatrix} w_1 & w_3 & w_5 \\ w_3 & w_1 & w_7 \\ w_6 & w_8 & w_2 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} w_1 & w_5 & w_7 \\ w_6 & w_2 & w_4 \\ w_8 & w_4 & w_2 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} w_1 & w_3 & w_5 & w_7 \\ w_3 & w_1 & w_7 & w_5 \\ w_6 & w_8 & w_2 & w_4 \\ w_8 & w_6 & w_4 & w_2 \end{vmatrix}.$$

The disorder solution for the nearest-neighbour model ($J = J' = J_4 = 0$) has been treated by Jaekel and Maillard (1985b). In this case we have

$$w_1w_2 = w_3w_4 = w_5w_6 = w_7w_8 = \mathrm{e}^{2K_0} \tag{10}$$

and F can be obtained directly from (5), by multiplying the first and last two equations respectively, yielding

$$F^2 = -4\mathrm{e}^{2K_0}\sinh 2K_1 \sinh 2K_2 \sinh 2K_2'/\sinh 2K_1'. \tag{11}$$

The condition (8) then implies that we are dealing with a frustrated model

$$K_1K_2K_1'K_2' < 0 \tag{12}$$

and (9) gives rise to an expression for the magnetic field L after substituting with (11). One also finds that x_i satisfies the quadratic equation

$$x_i^2 + A_ix_i + 1 = 0, \qquad i = 1, 2 \tag{13}$$

where

$$A_1 = [a^2 + (b + dF_1)^2 - (cF_1)^2]/a(b + dF_1)$$

$$a = b^{-1} = \exp(K_1 + K_2)$$

$$c = d^{-1} = \exp(-K_1' + K_2')$$

$$F_1 = \mathrm{sgn}(K_2')\left|\frac{\sinh 2K_1 \sinh 2K_2}{\sinh 2K_1' \sinh 2K_2'}\right|^{1/2}$$

and A_2 is given by a similar expression with K_2 and K_2' interchanged. The sufficient condition (8) can now be rewritten, using (13), as

$$\begin{aligned} &|d + (a+b)F_1^{-1}| \leq c \qquad &&\text{for } b + dF_1 > 0 \\ &|d + (a+b)F_1^{-1}| \geq c \qquad &&\text{for } b + dF_1 < 0 \end{aligned} \tag{14}$$

and similar expressions with K_2 and K_2' interchanged. The solution given by expressions (4) and (11) is valid if both of these conditions are satisfied.

In summary, we have studied the disorder solution for a general checkerboard Ising model in a field, and examined the difficulty associated with the change of boundary conditions, a step essential to the method of decimation. We obtained a sufficient criterion to ensure the validity of the change of boundary fields, and hence the decimation approach; we also obtained an equation for determining the disorder solution. The nearest-neighbour model considered by Jaekel and Maillard (1985b) has been re-examined in light of these discussions.

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